Cockcroft Institute

SPECIAL RELATIVITY

TUTORIAL PROBLEMS

Autumn Term 2012

For the following problems, the speed of light is $c = 2.99792458 \times 10^8\ m\ s^{-1}$*, the rest energy of an electron is 510.99891 keV, and the rest energy of a proton is 938.272031 MeV.*

1. Five street lamps A, B, C, D, E are located on a straight line along the x-axis equal distance 1 unit apart. They turn on at times $t_A = t_C = 1/c$, $t_B = 4/c$, $t_D = 2/c$, $t_E = 5/3c$ in the frame at rest relative to the ground. What is the order in which the light of the lamps reach the observer at $x = 0$ and compare this with the order in which they are switced on?

A car is moving at constant velocity $v = \frac{1}{3}c$ relative to the ground. At $t = t' = 0$ it is at $x = x' = 0$.

(a) Referring to space-time axes, what is the order in which the lamps turn on in the moving frame?

(b) What is the order in which the light from the lamps reach the observer in the car?

(c) Where is the car when light from street lamp D reaches it?

Illustrate these results on a space-time diagram.

2. In the coordinate system (ct, x), let A be the event $(0, 0)$ and B the event $(c\tau, v\tau + d)$. The worldline of an observer travelling at constant speed $v > 0$ is given by the equation $x = vt + d$, where $d > 0$ is a constant. Show that this observer regards the events A and B as simultaneous provided

$$\tau = \frac{d\,v}{(c^2 - v^2)}.$$

What is the distance between these events as measured by the observer?

3. Imagine you observe a train travelling past Warrington station at a relativistic speed v. Someone standing still on the train throws a ball in the direction the train is moving, with speed u. How fast do you observe the ball to be moving? What is the answer if $v = c$?

4. Before its closure, the Tevatron accelerator at Fermilab comprised five stages that sequentially boosted protons to 1 TeV (1000 GeV). Calculate the momentum and the proton speed at the end of each stage?

(i) Cockcroft-Walton pre-accelerator to 750 keV

(ii) Linac to 400 MeV

(iii) Booster to 8 GeV

(iv) Main injector to 150 GeV (express to at least five decimal places)

(v) Tevatron to 1 TeV (to at least seven decimal places)

5. A particle of rest mass m and four-momentum $\mathcal{P} = (E/c, p)$ is detected by an observer with four-velocity $\mathcal{V}$, satisfying $\mathcal{V} \cdot \mathcal{V} = c^2$.

Show that the speed of the detected particle in the observer's rest frame is

$$v = c\sqrt{1 - c^2 \frac{\mathcal{P} \cdot \mathcal{P}}{(\mathcal{P} \cdot \mathcal{V})^2}}.$$

6. A pion of rest mass M decays at rest into a muon of rest mass $m < M$ and a neutrino of zero rest mass. What is the speed u of the muon?

7. A moving π^0 particle of rest-mass m_π decays into two photons of zero rest-mass,

$$\pi^0 \to \gamma + \gamma.$$

Show that

$$\sin \tfrac{1}{2}\theta = \frac{m_\pi c^2}{2\sqrt{E_1 E_2}},$$

where θ is the angle between the 3-momenta of the two photons and E_1, E_2 are their energies.

8. A particle of rest mass M and total energy E collides with a particle of rest mass m at rest. Show that the sum E' of the total energies of the two particles in the frame in which their centre of mass is at rest is given by

$$E'^2 = (m^2 + M^2)c^4 + 2Emc^2.$$

9. A photon of energy E collides with a particle of rest mass m, which is at rest. The final state consists of a photon and a particle of rest mass M, $M > m$. Show that the minimum value of E for which it is possible for this reaction to take place is

$$E_{\text{min}} = \frac{M^2 - m^2}{2m} c^2.$$

10. An observer P moves with uniform acceleration a along the x-axis of the inertial frame of an observer O, starting from rest at $x = 0$ at $t = 0$. Show that the rapidity ρ of P with respect to O satisfies

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{a}{c},$$

where τ is proper-time for P. Show that the equation of the path of P in inertial coordinates for O is

$$ct = \frac{c^2}{a}\sinh\left(\frac{a\tau}{c}\right), \quad x = \frac{c^2}{a}\left(\cosh\left(\frac{a\tau}{c}\right) - 1\right),$$

where τ is the proper-time from rest in O.

A light signal is emitted by P at proper-time τ. Find the time at which it reaches the spatial origin. Hence or otherwise show that, if the light was emitted by P with frequency ω_e and observed at the origin with frequency ω_o, then

$$\omega_o = \omega_e \exp\left(-\frac{a\tau}{c}\right).$$